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GLOBAL SMOOTH SOLUTION OF CAUCHY PROBLEMS FOR A CLASS OF QUASILINEAR HYPERBOLIC SYSTEMS

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RESUME : Dans cet article, nous avons démontré que la présence de certains types de termes non homogènes peut rendre globalement régulière en temps la solution de systèmes hyperboliques quasilinéaires du premier ordre quand les conditions initiales sont assez petites.

ABSTRACT : In this article we proved that certain kinds of inhomogeneous terms can smoothen the solution for first order quasilinear hyperbolic systems globally in time provided that the initial data are small.

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ABSTRACT

In this article we prove that certain kinds of inhomogeneous terms can smoothen the solution for first order quasilinear hyperbolic systems globally in time provided that the initial data are small.

1. INTRODUCTION

For the Cauchy problem of first order quasilinear hyperbolic systems it is well-known that even if the initial data are very smooth, in general, a smooth solution exists only locally in time and singularities may appear in a finite time. It was discussed in [1] how the presence of various damping and dissipation mechanisms may influence the smoothness of the solution. In the same direction, we consider here the influence of a special kind of dissipative inhomogeneous term and establish the existence and uniqueness of global smooth solutions.

Consider the following Cauchy problem for a first order system of inhomogeneous hyperbolic balance laws:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} + g(u) = 0, & -\infty < x < \infty, \quad 0 \leq t < \infty & (1) \\ u(x,0) = \phi(x), & -\infty < x < \infty & (2) \end{cases}$$

where $u = (u_1,\dots,u_n)^T$ is the unknown vector function, $f(u)$ and $g(u)$ are given smooth vector functions of u with

$$g(0) = 0 \tag{3}$$

and $\phi(x)$ is smooth.

System (1) is hyperbolic on the domain under consideration if

1) The $n \times n$ matrix $\nabla f(u)$ has n real eigenvalues:

$$\lambda_1(u),\ \lambda_2(u),\dots,\lambda_n(u); \tag{4}$$

2) $\nabla f(u)$ is diagonalizable, i.e.,

$$\det \zeta(u) \neq 0. \tag{5}$$

where

$$\zeta(u) = \begin{pmatrix} \zeta_1(u) \\ \zeta_2(u) \\ \vdots \\ \zeta_n(u) \end{pmatrix} \tag{6}$$

and $\zeta_\ell(u) = (\zeta_{\ell 1}(u), \zeta_{\ell 2}(u), \ldots, \zeta_{\ell n}(u))$ is a left eigenvector corresponding to λ_ℓ:

$$\zeta_\ell(u)\nabla f(u) = \lambda_\ell(u)\zeta_\ell(u). \tag{7}$$

By (3), $g(u)$ can be rewritten as

$$g(u) = B(u)u \tag{8}$$

where $B(u) = (B_{ij}(u))$ is a $n \times n$ matrix with

$$B(0) = \nabla g(0). \tag{9}$$

Let
$$\lambda(u) = \mathrm{diag}\{\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)\}. \tag{10}$$

The main results in this paper are the following:

THEOREM 1. Assume that (3) holds and the matrix

$$\mathscr{A} \equiv (a_{ij}) = \zeta(0)\nabla g(0)\zeta^{-1}(0) \tag{11}$$

is row diagonal dominant:

$$a_{\ell\ell} > \sum_{j \neq \ell} |a_{\ell j}| \qquad (\ell = 1, \ldots, n) \tag{12}$$

Furthermore, let $\zeta(u), \lambda(u)$ and $B(u)$ be smooth with bounded C^1 norm and satisfy

$$|\det \zeta(u)| \geq D_0 > 0 \qquad (D_o \text{ is constant}). \tag{13}$$

Then the Cauchy problem (1), (2) admits a unique global smooth solution $u = u(x,t)$ on $t \geq 0$ which decays exponentially with respect to t, provided that the C^1norm of the initial data $\phi(x)$ is sufficiently small.

THEOREM 2. The conclusion of Theorem 1 holds if (12) is replaced by the following weaker condition that there exists a diagonal matrix γ. with $\gamma_\ell \neq 0$ ($\ell=1,\ldots,n$) such that the matrix $\gamma\mathscr{A}\gamma^{-1}$ is row diagonal dominant.

The results in this paper generalize and make more precise the results in [4] for the one-dimensional case.

We should note that our assumption (12) or even (15) do not cover, for example, the case of the one-dimensional damped wave equation

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x + v = 0. \end{cases} \tag{14}$$

For a global existence theorem for this special system see [5].

The proofs of Theorem 1 and 2 will be given in the next section.

2. PROOFS OF THEOREMS

Multiplying (1) by ζ from the left and using (7), we obtain the characteristic form

$$\zeta(u)[\frac{\partial u}{\partial t} + \lambda(u) \frac{\partial u}{\partial x}] + A(u)\zeta(u)u = 0, \tag{15}$$

where

$$A(u) \equiv (A_{ij}(u)) = \zeta(u)B(u)\zeta^{-1}(u). \tag{16}$$

We note, in view of (9), that $A(0) = \mathscr{A}$, i.e., $A_{ij}(0) = a_{ij}$ $(i,j=1,\ldots,n)$. It is now clear that the assertion of Theorem 1 follows easily from the following lemma.

LEMMA. Assume that the matrix $A(0) = \mathscr{A}$ satisfies (12) and that $\zeta(u),\lambda(u)$ and $A(u)$ are smooth with bounded C^1norms and satisfy (13). Then, if the C^1 norm of the initial data $\phi(x)$ is sufficiently small, there exists a unique global smooth solution $u = u(x,t)$ on $t \geq 0$ for the Cauchy problem (15), (2) and this solution decays exponentially as $t \to \infty$.

PROOF. In order to obtain the existence of the global smooth solution it suffices to prove that if the initial data $\phi(x)$ have a small C^1norm then the C^1norm of the smooth solution $u = u(x,t)$ defined on the domain

$$R(T) = \{(t,x) \; 0 \leq t \leq T, \; -\infty < x < \infty\} \tag{17}$$

does not depend on T.

Rewrite (15) in the form

$$\sum_j \zeta_{\ell j}(u)\left(\frac{\partial u_j}{\partial t} + \lambda_\ell(u)\frac{\partial u_j}{\partial x}\right) + \sum_j (A_{\ell j}(u)-A_{\ell j}(0)) \sum_k \zeta_{jk}(u)u_k +$$

$$\sum_j a_{\ell j} \sum_k \zeta_{jk}(u)u_k = 0 \qquad (\ell=1,\ldots,n) \tag{18}$$

Let

$$\xi = f_\ell(\tau;t,x) \tag{19}$$

be the ℓ-th characteristic curve passing through the point (t,x), which satisfies

$$\begin{cases} \dfrac{d\xi}{d\tau} = \lambda_\ell(\tau, f_\ell(\tau;t,x)), \\ f_\ell(t;t,x) = x. \end{cases} \tag{20}$$

Setting

$$U_\ell = \sum_k \zeta_{\ell k}(u)u_k \tag{21}$$

and

$$V_\ell = e^{a_{\ell\ell}t} U_\ell, \tag{22}$$

it is easy to see that

$$\begin{cases} \dfrac{\partial V_\ell}{\partial t} + \lambda_\ell(u)\dfrac{\partial V_\ell}{\partial x} = e^{a_{\ell\ell}t}\{\sum_j[\dfrac{\partial\zeta_{\ell j}(u)}{\partial t} + \lambda_\ell(u)\dfrac{\partial\zeta_{\ell j}(u)}{\partial x}]u_j \\ \qquad -\sum_j[A_{\ell j}(u)-A_{\ell j}(0)]\sum_k\zeta_{jk}(u)u_k\} - e^{a_{\ell\ell}t}\sum_{j\neq\ell}a_{\ell j}U_j \\ \qquad\qquad (\ell=1,\dots,n), \\ V_\ell = U_\ell^0(x) \equiv \sum_k \zeta_{\ell k}(\phi)\phi_k, \quad t=0, \quad -\infty < x < \infty. \end{cases} \tag{23}$$

Integrating the ℓ-th equation of (23) along the ℓ-th characteristic curve, we get

$$V_\ell(t,x) = U_\ell^0(f_\ell(0;t,x)) + \int_0^t\{e^{a_{\ell\ell}\tau}[\sum_j(\frac{\partial\zeta_{\ell j}(u)}{\partial t} + \lambda_\ell(u)\frac{\partial\zeta_{\ell j}(u)}{\partial x})u_j - \sum_j(A_{\ell j}(u)-A_{\ell j}(0))\sum_k\zeta_{jk}(u)u_k] - e^{a_{\ell\ell}\tau}\sum_{j\neq\ell}a_{\ell j}U_j\}d\tau, \tag{24}$$

then

$$U_\ell(t,x) = e^{-a_{\ell\ell}t}U_\ell^0(f_\ell(0;t,x)) + \int_0^t \{e^{a_{\ell\ell}(\tau-t)}[\sum_j(\frac{\partial\zeta_{\ell j}(u)}{\partial t} + \lambda_\ell(u)\frac{\partial\zeta_{\ell j}(u)}{\partial x})u_j$$

$$- \sum_j(A_{\ell j}(u)-A_{\ell j}(0))\sum_k\zeta_{jk}(u)u_k] - e^{a_{\ell\ell}(\tau-t)}\sum_{j\neq\ell}a_{\ell j}U_j\}d\tau. \tag{25}$$

in which the integration is carried along the characteristic $(\tau,f_\ell(\tau;t,x))$.

We have

$$\frac{\partial\zeta_{\ell j}(u)}{\partial t} + \lambda_\ell(u)\frac{\partial\zeta_{\ell j}(u)}{\partial x} = \sum_k\frac{\partial\zeta_{\ell j}(u)}{\partial u_k}(\frac{\partial u_k}{\partial t} + \lambda_\ell(u)\frac{\partial u_k}{\partial x}). \tag{26}$$

System (18) together with (21) imply that

$$\frac{\partial u_k}{\partial t} = \sum_{p,m}\zeta^{k,p}(u)\zeta_{p,m}(u)\lambda_p(u)\frac{\partial u_m}{\partial x} - \sum_{p,m}\zeta^{k,p}(u)A_{p,m}(u)U_m, \tag{27}$$

where $\zeta^{k,p}(k,p = 1,\dots,n)$ denote the elements of ζ^{-1}. Thus, introducing

$$w_i = \frac{\partial u_i}{\partial x}, \quad \overline{W}_\ell = \sum_j\zeta_{\ell j}(u)w j, \tag{28}$$

it follows from (26) and (27) that

$$\begin{cases} \frac{\partial\zeta_{\ell j}(u)}{\partial t} + \lambda_\ell(u)\frac{\partial\zeta_{\ell j}(u)}{\partial x} = \sum_s P_{\ell js}(u)\overline{W}_s + \sum_{p,s}Q_{\ell jp}(u)A_{p,s}(u)U_s, \\ \frac{\partial u_k}{\partial t} = \sum_s R_{k,s}(u)\overline{W}_s + \sum_{p,s}S_{k,p}(u)A_{p,s}(u)U_s, \end{cases} \tag{29}$$

in which

$$\begin{cases} P_{\ell js}(u) = \sum_{k,p,m} \dfrac{\partial\zeta_{\ell j}(u)}{\partial u_k} \zeta^{k,p}(u)\zeta_{p,m}(u)\lambda_p(u)\zeta^{ms}(u) + \sum_k \dfrac{\partial\zeta_{\ell j}(u)}{\partial u_k}\lambda_\ell(u)\zeta^{ks}(u), \\ Q_{\ell jp}(u) = -\sum_k \dfrac{\partial\zeta_{\ell j}(u)}{\partial u_k}\zeta^{kp}(u), \\ R_{ks}(u) = \sum_{p,m} \zeta^{kp}(u)\zeta_{p,m}(u)\lambda_p(u)\zeta^{ms}(u), \\ S_{kp}(u) = -\zeta^{kp}(u). \end{cases} \tag{30}$$

Then, from (25) we have

$$U_\ell(t,x) = e^{-a_{\ell\ell}t} U_\ell^0(f_\ell(0;t,x)) +$$

$$\int_0^t \{e^{a_{\ell\ell}(\tau-t)} [\sum_j (\sum_s P_{\ell js}(u)\overline{W}_s + \sum_{p,s} Q_{\ell jp}(u)A_{ps}(u)U_s)\cdot \sum_r \zeta^{jr}(u)U_r \tag{31}$$

$$- \sum_j (A_{\ell j}(u)-A_{\ell j}(0))U_j] - e^{a_{\ell\ell}(\tau-t)} \sum_{j\neq\ell} a_{\ell j}U_j\}d\tau.$$

Moreover, if we set

$$H_\ell = e^{a_{\ell\ell}t}\overline{W}_\ell \tag{32}$$

and differentiate (18) with respect to x, it follows in a similar way that

$$\begin{cases} \dfrac{\partial H_\ell}{\partial t} + \lambda_\ell(u)\dfrac{\partial H_\ell}{\partial x} = e^{a_{\ell\ell}t}\cdot\sum_j\left(\dfrac{\partial\zeta_{\ell j}(u)}{\partial t} + \lambda_\ell(u)\dfrac{\partial\zeta_{\ell j}(u)}{\partial x}\right. w_j \\ \quad - e^{a_{\ell\ell}t}\left(\sum_{j,k}\dfrac{\partial\zeta_{\ell j}}{\partial u_k}\cdot\dfrac{\partial u_j}{\partial t}w_k + \sum_{j,k}\dfrac{\partial(\zeta_{\ell j}\lambda_\ell)}{\partial u_k}w_jw_k + \sum_{j,k,p}\dfrac{\partial(A_{\ell j}\zeta_{jk})}{\partial u_p}w_pu_k\right. \\ \quad + \sum_j (A_{\ell j}(u)-A_{\ell j}(0))\overline{W}_j) - e^{a_{\ell\ell}t}\sum_{j\neq\ell}a_{\ell j}\overline{W}_j, \\ H = \overline{W}_\ell^0(x) \equiv \sum_k \zeta_{\ell k}(\phi)\phi_k'(x), \quad t = 0, \quad -\infty < x < \infty. \end{cases} \tag{33}$$

Integrating the ℓ-th equation along the ℓ-th characteristic curve and using (29), we obtain

$$\begin{aligned} W_\ell(t,x) = {} & e^{-a_{\ell\ell}t}\overline{W}^0_\ell(f_\ell(0;t,x)) \\ & + \int_0^t \{e^{a_{\ell\ell}(\tau-t)} [\sum_j (\sum_s P_{\ell js}(u)\overline{W}_s + \sum_{p,s} Q_{\ell jp}(u)A_{ps}(u)U_s)\cdot\sum_r \zeta^{jr}\overline{W}_r \\ & - \sum_{j,k} \frac{\partial\zeta_{\ell j}}{\partial u_k}(\sum_s R_{ks}(u)\overline{W}_s + \sum_{p,s} S_{kp}(u)A_{ps}(u)U_s)\cdot\sum_r \zeta^{kr}\overline{W}_r \\ & - \sum_{j,k,s,r} \frac{\partial(\zeta_{\ell j}\lambda_\ell)}{\partial u_k}\zeta^{js}\zeta^{kr}\overline{W}_s\overline{W}_r - \sum_{j,k,p,r,s} \frac{\partial(A_{\ell j}\zeta_{jk})}{\partial u_p}\zeta^{kr}\zeta^{ks}\overline{W}_r U_s \\ & - \sum_j (A_{\ell j}(u)-A_{\ell j}(0))\overline{W}_j] - e^{a_{\ell\ell}(\tau-t)}\sum_{j\neq\ell} a_{\ell j}\overline{W}_j\}d\tau. \end{aligned} \tag{34}$$

Let

$$U_\ell(t) = \sup_{\substack{0\le\tau\le t \\ -\infty<x<\infty}} |U_\ell(\tau,x)|, \qquad U(t) = \sup_\ell U_\ell(t), \tag{35}$$

$$\overline{W}_\ell(t) = \sup_{\substack{0\le\tau\le t \\ -\infty<x<\infty}} |\overline{W}_\ell(\tau,x)|, \qquad \overline{W}(t) = \sup_\ell \overline{W}_\ell(t), \tag{36}$$

and

$$a = \min_\ell\{a_{\ell\ell}\}. \tag{37}$$

We can easily obtain from (31) that

$$\begin{aligned} U_\ell(t) \le e^{-at}C_0 + \int_0^t \{ & D_1 e^{a(\tau-t)}[\overline{W}(\tau)U(\tau)+U^2(\tau)] + \\ & e^{a_{\ell\ell}(\tau-t)}\sum_{j\neq\ell}|a_{\ell j}|U_j(\tau)\}d\tau, \end{aligned} \tag{38}$$

where

$$C_0 = \sup_{\substack{\ell \\ -\infty < x < \infty}} |U_\ell^0(x)| = \sup_{\substack{\ell \\ -\infty < x < \infty}} |\sum_k \zeta_{\ell k}(\phi)\phi_k| \tag{39}$$

and $D_i (i=1,2,\ldots)$ will denote throughout various fixed constants.

Since

$$\int_0^t e^{a_{\ell\ell}(\tau-t)} d\tau = \frac{1}{a_{\ell\ell}} (1-e^{-a_{\ell\ell}t}),$$

it follows on account of (12),

$$\begin{aligned} &\int_0^t e^{a_{\ell\ell}(\tau-t)} \sum_{j\neq\ell} |a_{\ell j}| U_j(\tau) d\tau \\ &\leq \int_0^t e^{a_{\ell\ell}(\tau-t)} \sum_{j\neq\ell} |a_{\ell j}| d\tau \cdot U(t) \\ &\leq \frac{\sum_{j\neq\ell} |a_{\ell j}|}{a_{\ell\ell}} U(t) \leq dU(t), \end{aligned} \tag{40}$$

where

$$d = \max_\ell \frac{\sum_{j\neq\ell} |a_{\ell j}|}{a_{\ell\ell}} < 1. \tag{41}$$

Then (38) gives

$$U(t) \leq \frac{1}{1-d} e^{-at} C_0 + \int_0^t D_2 e^{a(\tau-t)} (\overline{W}(\tau)U(\tau)+U^2(\tau)) d\tau. \tag{42}$$

In a similar way (34) gives

$$\left\{ \begin{aligned} \overline{W}_\ell(t) \leq e^{-at} C_1 + \int_0^t \{ & D_3 e^{a(\tau-t)} (\overline{W}^2(\tau)+\overline{W}(\tau)U(\tau)) . \\ & -e^{a_{\ell\ell}(\tau-t)} \sum_{j\neq\ell} |a_{\ell j}| \overline{W}_j(\tau) \} d\tau \end{aligned} \right. \tag{43}$$

where

$$C_1 = \sup_{\substack{\ell \\ -\infty < x < \infty}} |\overline{W}^0_\ell(x)| = \sup_{\substack{\ell \\ -\infty < x < \infty}} |\sum_k \zeta_{\ell k}(\phi)\phi_k'|. \tag{44}$$

Therefore,

$$\overline{W}(t) \leq \frac{1}{1-d} e^{-at} C_1 + \int_0^t D_4 e^{a(\tau-t)} (\overline{W}^2(\tau) + \overline{W}(\tau)U(\tau))d\tau. \tag{45}$$

Let

$$X(t) = U(t) + \overline{W}(t). \tag{46}$$

Combining (42) and (45), it is easily seen that

$$X(t) \leq \frac{1}{1-d} e^{-at}(C_0+C_1) + \int_0^t D_5 e^{a(\tau-t)} X^2(\tau)d\tau. \tag{47}$$

Letting

$$Y(t) = X(t)e^{at}, \tag{48}$$

(47) can be written as

$$Y(t) \leq \frac{1}{1-d} (C_0+C_1) + \int_0^t D_5 e^{-a\tau} Y^2(\tau)d\tau. \tag{49}$$

Consider

$$\begin{cases} \dfrac{dZ(t)}{dt} = D_5 e^{-at} Z^2(t), \\ Z = \dfrac{1}{1-d}(C_0+C_1) \quad , \quad t = 0. \end{cases} \tag{50}$$

We then have the estimation

$$Y(t) \leq Z(t). \tag{51}$$

But

$$Z(t) = \frac{1}{\frac{1-d}{C_0+C_1} - \frac{D_5}{a}(1-e^{-at})} . \tag{52}$$

Therefore, if $C_0 + C_1$ is so small that

$$a > D_5 \cdot \frac{C_0+C_1}{1-d} , \tag{53}$$

then

$$Z(t) \le \frac{C_0+C_1}{1-d} . \tag{54}$$

Hence

$$X(t) \le \frac{C_0+C_1}{1-d} e^{-at}. \tag{55}$$

Thus, by the definition of $X(t)$, the lemma and thenceforth Theorem 1, is proved. ∇∇∇

REMARK 1. It follows from the proof of the theorem that assumption (13) of Theorem 1 can be replaced with the weaker

$$\det \zeta(u) \neq 0. \qquad (56$$

REMARK 2. Assumption (12) in Theorem 1 is similar to the assumption in [2] for establishing the corresponding global existence of discontinuous solutions for the Cauchy problem (1),(2).

By means of the method in [3], multiplying the ℓ-th equation of (18) by a real number $\gamma_\ell \neq 0$, the matrix $\mathscr{A}$ can be replaced with $\gamma\mathscr{A}\gamma^{-1}$, where

$$\gamma = \mathrm{diag}\{\gamma_1,\gamma_2,\ldots,\gamma_n\}. \tag{57}$$

Thus Theorem 2 follows from Theorem 1.

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